Numbers

Juhan Puhm

Numbers

Contents

	Page
Chapter 1 ~ Natural Numbers	
-Natural Numbers	3
-Notes on the Words Infinite and Infinity	6
-Mathematical Operators	9
-Properties of Natural Numbers	19
-Mathematical Induction	23
-Conclusion	30
Chapter 2 ~ Integers	
-Integers	31
-Positive and Negative Symbols	32
-Mathematical Operators	34
Chapter 3 ~ Rational Numbers	xx
Chapter 4 ~ Irrational Numbers	xx
Chapter 5 ~ Complex Numbers	xx

Chapter 1

Natural Numbers [**N**] (0), 1, 2, 3, 4, 5…
Whole Numbers [**W**] 0, 1, 2, 3, 4, 5…

~ **Natural Numbers** can be simply defined as "counting numbers".
~ Zero may or may not be included in the definition of Natural Numbers.
~ Natural Numbers that include zero may also be called **Whole Numbers**.

*Our definition of **Natural Numbers** will include zero, as it is ridiculous to not know what happened to one apple after we ate it!*

~~~~~

Types of Natural Numbers

Cardinal Numbers are "counting numbers" and tell us the quantity, or how many of something we have, for example: 6 coins or 0 apples.
-From Latin *cardinalis* (adjective), meaning "fundamental, most important, principal, chief, essential, that on which something turns".

Ordinal Numbers tell us the "position or rank" of something like first, second, third, nth, etc.
-From Latin o*rdinalis* (adjective), meaning "the place or position of an object in a row, series, succession, or order".

Nominal Numbers do not include quantity or position. They are simply used as a "label" to uniquely identify or name something. Examples include things like a bus route or sports jersey. A postal code can use both nominal numbers and nominal letters.
-From Latin *nominalis* (adjective), meaning "belonging or pertaining to a name".

~~~~~

Symbols

The symbol **N** is used to symbolize the set of natural numbers, or **W** for whole numbers.

Symbols called "numbers" are used to represent the quantity, position, or name of something.
There are many symbolic ways to represent natural numbers.
For example, if we have fifty-three of something: 53, 110101, LIII.

~~~~~

Finite Symbol

All natural numbers are **Finite**.

The word **finite** comes from the Latin word ***finitus*** which means "finished".

We can define the word **finite** (adjective) in a number of different ways:
1) "having a measurable or definable limit".
2) "having bounds or limits" or "limited in size or extent".

The word **finite** is an adjective. Adjectives describe specific features or attributes of a noun, for example, a *large red ripe* apple, or *finite* time, area, resource, number, etc.

We will start with the first definition of **finite**, which is "having a measurable or definable limit". Another way of saying something is "*measurable*" or "*definable*" is to say that it is ***countable***.

Starting from zero, natural numbers can be counted forever without end. No matter how big they get, natural numbers never magically cease to be countable, measurable, or **finite**.

At no point do *individual* **finite natural numbers** magically become **in-finite**, meaning *"not"* **finite**, or simply, not measurable, definable, or ***countable***.

The (positive) **Finite** symbol $+\propto$ (open lemniscate) represents numbers that endlessly get larger without ever ceasing to be **finite**.

[The common term used here is "*potential infinity*", which is unfortunate, as endlessly countable, **finite** natural numbers do not have the *potential* to ever reach *infinity*. The word "*infinity*" is a noun, which means it must be either a person, a place, thing, idea, etc. We will look at the words **Infinite** and **Infinity** further on.]

~~~~~

*We can for fun create the adjective "**ad-finite**" for a finite number getting larger forever!*
The prefix "**ad-**" denotes motion or direction, such as *advance*. It also denotes addition or increase, such as *adjunct,* or even *ad(d)-ing, (get it!).*

An "**adfinite"** number that gets larger forever will never cease to have a **finite** value that can be added to, subtracted from, multiplied or divided, compared and so on.

"**Infinity**" (noun) then is not a place that a **finite** number magically arrives "*to"* at its journey's end, and magically ceases to be countable, measurable, or **finite**.

An "**adfinite"** number doesn't *"go to"* a place called "**infinity"** (noun), it instead: *"goes on"* "**ad finitum**" (*an adverb describing how the phrasal verb "goes on" proceeds*), meaning it gets larger endlessly and never ceases to be **finite**, (*not* ad infinitum!).

If we want, we can say an "**adfinite"** number "*goes to*" "**adfinity**" (noun), meaning to the idea of a very large but **finite** arbitrary number that can still be measured.

We can write then (using the positive **Finite** or **Adfinity** symbol):

Natural Numbers [**N**] = 0, 1, 2, 3, 4, 5…+∝

Two properties of **Natural Numbers** then are:
1) individually they can never cease to be **finite**.
2) that they can increase in magnitude endlessly.

~~~~~

Conclusion

There is a duality between *individual* **finite natural numbers**, and **finite natural numbers** *as a whole.*

Finite natural numbers are all *"**countable**"*. Any *individual* **finite natural number** we choose automatically has a "*bound, limit*, *size* or *extent"* and is "*measurable or definable"*.

Finite natural numbers however are *"**countable**"* forever. Only *as a whole* can they be said to be without "*bounds, limit*, *size* or *extent".* It is not possible to *"measure or define"* the largest **finite natural number** as they are without end. Only *as a whole* can **finite natural numbers** be said to be **in-finite**, meaning without "*bounds, limit*, *size, extent"*, and without a *"measurable or definable limit".*

Expanding our opening definition:
~**Natural numbers** can be simply defined as **finite** "counting numbers", which *individually* can be "counted" forever, and as a *whole* are endless.
~~~~~

Notes on the words "Infinite" and "Infinity"

We can define the word **in-finite** as that which is *not* **finite**.

From the first definition of the word **finite**: the word **in-finite** then is that which has *no "measurable or definable limit"*. Another way of saying something is *not* "*measurable or definable*" is to say that it is *not* ***countable***. We will further return to this definition when we look at irrational numbers.

From the second definition of the word **finite**: the word **in-finite** then is that which has *no "bounds or limits"* or is *not "limited in size or extent"*. This is the common usage of the adjective **infinite**.

~~~~~

Let us look carefully at the second definition of **in-finite**, which is having *no "bounds or limits"* or *not "limited in size or extent"*. The meaning of this definition is subtly different from the first definition of **in-finite**, which is having *no "measurable or definable limit".*

We commonly say that endless space is **infinite**, meaning that it goes on forever without *end* or *bounds*. An **infinite** universe means that there is no *limit* to its *extent* or *size*. Spatially, an **infinite** universe is endless. In the same way, natural numbers have no end and can go on forever, and so natural numbers, (not *individually* but) *as a whole*, can be said to be **infinite**.

But, existing in three-dimensional space as we do, we can only *measure* space or distance in a **finite** way, wherever we are, or whatever we are looking at. In the same way, *individual* natural numbers never cease to be **finite**, in that they are always *measurable, definable,* and *countable*.

This is a most interesting duality. Though space and numbers can go on forever, and *as a whole* are **infinite**, without "*bounds, limit*, *size* or *extent"*, we can only ever *measure* them in a **finite** way as "*having a measurable or definable limit*" !

~~~~~

The main difficulty is with the word **infinity** as a *noun* (person, place, thing, idea, etc.), as it implies that there is an actual place called **infinity**, for example poetically, "the clouds stretched away *to* infinity". However, we can never with a **finite** process, like counting or going out into space, arrive at a place or thing called **infinity**!

Simply then, there is no such place or thing in a **finite** world as **infinity**, except poetically, imaginatively, figuratively, or as an *idea*. The *idea* of **infinity** is fanciful, vague and mystical, as the word **infinity** as a *noun* implies there is such a “place” or “thing” as **infinity** in existence in a **finite** world!

~~~~~

One given definition of **infinity** (noun) is the “quality or state of being **infinite** (adjective)”. This definition is difficult to make sense of grammatically and logically. It is the same as saying that a noun has a “*quality or state*” of being an adjective. **Finite** natural numbers and space, only *as a whole*, have the “*quality or state*” of being **infinite** (adjective), only because they “*go on*” forever. *Individual* **finite** numbers and *measurable* space do not “*go to”* and arrive finally at some mystical “*quality or state*” of **infinity** (noun).

Infinity should then instead be defined as having the “*quality or state”* of endlessly “*going on”*, meaning getting larger or more distant forever! This would make the word **infinity** closer to a verb, or a stative verb, which describes a “*state of being or condition*”. **Infinity** then would have the “*quality* or *state”* of endlessly getting larger or bigger. But this continual process can also be thought of as an action verb like *adding* or *expanding*!

~~~~~

Another definition of **infinity** is a “number greater than any assignable quantity or countable number”. This definition makes even less sense, or no sense at all.

This definition does not apply to the second definition of **infinite**, as being without “*bounds, limits, size* or *extent”. Individual* numbers are **finite.** There is no greatest *number*, and no *number* that is *boundless, limitless,* or *endless*. For a *number t*o be “*greater*” than “*any assignable quantity or countable number*” makes no sense.

It also doesn’t apply to the first definition of **infinite**, as being not “*measurable or definable”,* or not *“countable”*. If something is *not* measurable, definable, or countable, then the terms “greater than” or “lesser than” make no sense and cannot apply. The only thing “*greater”* than “*any assignable quantity or countable number*” is another **finite** number!

The word “*any*” in the phrase “*greater than any assignable quantity or countable number*” is an attempt at bridging the **finite** and the **infinite**, but in the end this makes no logical sense and requires a leap of faith, or in other words, a mystical point where **finite** numbers (or space) magically reach **infinity**!

If **infinite** is used as a noun it is always proceeded by the definite article “the”, as in “**the infinite**”. It is hard to think that there is any difference between “the infinite" and "infinity” conceptually.

~~~~~

Summary

We must always then, *measure* numbers and space in a **finite** way. Only as a *whole* can we say numbers and space are **Infinite** (adjective), as in **infinite** numbers and **infinite** space.

Individual numbers and *measurable* space endlessly *"go on"* **ad finitum** (*not* ad infinitum), never ceasing to be **finite**.

The word **Infinity** (noun) is problematic as it implies **infinity** is a place or a thing. Instead of a noun, **infinity** is better thought of as a *stative or action* verb, having the "*quality or state*" of *"going on"* forever.

As a "*transcendent idea",* **infinity** can be said to be ***"meta-finite"***, a mystical and fancy way of saying "*beyond*" the **finite**. A "*transcendent idea"* is an idea that "*transcends, goes beyond, exceeds, is above, surpasses, is superior to",* ordinary or common "*ideas, thoughts, concepts, limits, experience, understanding, perceptions*", *(whether grounded in reality or not, depending upon one's idea of reality!)*.

We look at **Actual Infinity** carefully in the essay **"Plotting 'e' & Ideas of Infinity"**.
~~~~~

Mathematical Operators

We will look at six mathematical operations in three related pairs:
~ Addition and Subtraction
~ Multiplication and Division
~ Exponents and Roots

~~~~~
### Addition and Subtraction

**Natural numbers** may be added to and subtracted from one another.
Example: 5 + 2 = 7, 9 - 9 = 0

When subtracting natural numbers, for example $a – b = c$, $a$ must be greater than or equal to $b$ ($a \geq b$), as natural numbers can't be negative.

~~~~~
Multiplication

If the same number is added many times together the short form multiplication symbol "**x**" (called "*times*") may be used. (The symbol ***** is also used for multiplication)

For multiplication we are seeking the *sum* of a number added to itself a number of *"times"*.

Example: 5 + 5 + 5 + 5 = 5 **x** 4 (meaning 5 added together 4 *times*) = 20
4 + 4 + 4 + 4 + 4 = 4 **x** 5 (meaning 4 added together 5 *times*) = 20

The value or quantity to be multiplied is called the **multiplicand**.

The number determining how many *times* the multiplicand is added together is called the **multiplier**.

The term **product** is used to define the result of one or more multiplications.

Therefore, the **multiplicand** *"times"* the **multiplier** equals the **product.**

This implies for example, that 0 x 7 = 0 + 0 + 0 + 0 + 0 + 0 + 0 which equals 0. Here 0 is added together 7 *"times".*

It is harder to understand, for example, what 7 x 0 is? Here 7 appears 0 *"times",* which implies there are ***no*** 7's. It is like specifically pointing out the ***absence*** of any 7's.

0 x 0 then is the "***absence*** of 0" which would be "the void".
(0 x 0) x 0 then is the ***absence*** of "the void",
or in other words, the "***absence*** of the (***absence*** of 0)"!

~~~~~

Division

Dividing a number into a specific number of *equal parts* is a more complicated process.
We are seeking the size of an *equal part* of a number, the size of the *equal part* being dependent on how many *equal parts* the number has been divided into.

Example: 20 / 4 = 5 (means 20 divided into 4 *equal parts* each of size 5) as 20 = 5 + 5 + 5 + 5
20 / 5 = 4 (means 20 divided into 5 *equal parts* each of size 4) as 20 = 4 + 4 + 4 + 4 + 4

The term **factor** is used to define all the divisors of a number (that leave no remainder).

For example, the factors of 64 are 1, 2, 4, 8, 16, 32 and 64.

When one natural number can't be divided completely into *equal parts* by another natural number then the concept of a **remainder** can be used.

Example: 17 / 3 = 5 remainder 2

The number being divided (17) is called the **dividend**.
The number doing the dividing (3) is called the **divisor**.
The quantity or natural number part of the answer (5) is called the **quotient**.
The fractional part of the answer (2) is called the **remainder**.

~~~~~

We can also think of **division** as a kind of "inversion" of **multiplication.**

For **multiplication**:

multiplicand x **multiplier** = **product**

For **division** this can be rearranged as either:

1) **product / multiplier = multiplicand**, or

2) **product / multiplicand = multiplier**

The equivalent terms for multiplication and division would then be:

multiplicand = quotient
multiplier = divisor
product = dividend

And so, the ***division*** equivalent of 1) would then be:

1) **dividend / divisor = quotient**

This is the definition of **division** we have above where:

a) we start with a number (***dividend***)
b) we divide the number into a number of *equal* parts (***divisor***), and
c) we find the size of the *equal* parts (***quotient***)
d) (whatever is left over we call the ***remainder***)

The ***division*** equivalent of 2) is different:

2) **dividend / quotient = divisor**

where:

a) we start with a number (***dividend***)
b) we take the size of an *equal part* of the number (***quotient***), and
c) we find how many times the *equal part* fits into the number (***divisor***)
d) (whatever is left over we call the ***remainder***)

Though 1) and 2) are different, when ***dividing*,** we rarely (if ever) make the possible distinction between dividing the ***dividend*** by the ***divisor*** or dividing the ***dividend*** by the ***quotient***.

In the same way, when ***multiplying***, we rarely (if ever) make the distinction between which number is the ***multiplicand*** and which number is the ***multiplier***.

~~~~~

With ***multiplication***, we <u>*add*</u> a number to itself a number of "*times*" to find a *sum.*

In the same way, with ***division***, we can <u>*subtract*</u> a number from a *total* a number of "*times*" and seek to find how many "*times*" we can do this to reach 0.

Again, for ***multiplication*** we are seeking the *sum* of a number <u>*added*</u> to itself a number of *"times"*.

Example: 5 + 5 + 5 + 5 = 5 **x** 4 (meaning 5 added together 4 *times*) = 20
4 + 4 + 4 + 4 + 4 = 4 **x** 5 (meaning 4 added together 5 *times*) = 20

For ***division***, for the second example 2) above, we are seeking the number of "*times*" an *equal part* can be <u>*subtracted*</u> from a *total*. Or in other words, from a *total* (***dividend***), we can <u>*subtract*</u> an *equal part* (***quotient***)***,*** how many *"times" (**divisor**)?*

Example: 20 / 4 = 5 (then means from 20 the *equal part* 4 is subtracted 5 *times*)
as 20 -4 -4 -4 -4 -4 = 0
20 / 5 = 4 (then means from 20 the *equal part* 5 is subtracted 4 *times*)
as 20 -5 -5 -5 -5 = 0

~~~~~

<u>Summary of Multiplication and Division</u>

Multiplication is a number added to itself one or many times seeking a total value or *sum*.

Division is a number divided into one or many *equal parts* seeking the size of the *equal part*.

Division can also be, from a *total* subtracting an *equal part* a number of times, and seeking to find how many *times* we can subtract the *equal part* from the *total.*

All **natural numbers** can be added to themselves any number of times, or in other words multiplied together, always giving another natural number.

Example: 4 + 4 + 4 = 4 x 3 = 12, 7 + 7 + 7 + 7 = 7 x 4 = 28

All **natural numbers** can be divided into at least one **natural number** part when divided by 1, but not all **natural number** divisions of a **natural number** into equal parts will result in a part size that is also a **natural number**.

Example: 15 / 1 = 15, 15 / 3 = 5, 15 / 4 ≠ natural number

We look at division by **0** and **Actual Infinity** in the essay **“Plotting ‘e’ & Ideas of Infinity”**.

~~~~~

Exponents

An exponent is a special type of multiplication where a number is *multiplied* by itself one or more times. An exponent is the short form of this specific type of multiplication, multiplication being a short form for addition. An exponent then can also be reduced to addition.

To **square** a number is to multiply a number by itself twice. It is called a square because the sides of a square have the same length. When two perpendicular sides of a square are multiplied together they give the area of the square. The unit of area then is always physically a square. This will prove endlessly problematic.
Example: 5^2 = 5 x 5 = 5 + 5 + 5 + 5 + 5 = 25

To **cube** a number is to multiply a number by itself three times. It is called a cube because the sides of a cube all have the same length. When three perpendicular sides of a cube are multiplied together they give the volume of the cube. The unit of volume then is always physically a cube. This will prove endlessly problematic.
Example: 3^3 = (3 x 3) x 3 = (3 + 3 + 3) + (3 + 3 + 3) + (3 + 3 + 3) = 27
4^3 = (4 x 4) x 4 = (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) + (4 + 4 + 4 + 4) = 64

We can continue raising any natural number to any natural number exponent. The result will always be a natural number.

We can see how a **square** exponent is analogous to 2-dimensional space, giving us the area of a square based on the length of its equal sides.
~~~~~

We can see how a **cube** exponent is analogous to 3-dimensional space, giving us the volume of a cube based on the length of its equal sides.

It makes sense then that raising a number to a higher natural number exponent is analogous to higher number dimensions, even though they are impossible to visualize in the reality of our 3-dimensional space.

It makes sense then that a number raised to exponent 1 is analogous to 1-dimensional space, giving us the length of a **line.**
Example: $5^1 = 5$

It makes sense then that a number raised to exponent 0 is analogous to 0-dimensional space, which is a **point**. A point has neither length, breadth, or height, but yet still exists, at least in our thoughts, conceptually or by inference. By virtue of existence by thought, a number raised to the exponent 0 is equal to 1.

Example: $4^0 = 1$

We can however verify that a number to the exponent 0 is equal to 1 in a different, more mathematical way.

When we multiply two same numbers together that both have exponents (called having the same base), a shortcut is to simply add the exponents together.
Example: $3^3 \times 3^2 = (3 \times 3 \times 3) \times (3 \times 3) = 3^{(3+2)} = 3^5$

When we divide two same numbers together that both have exponents, a shortcut is to simply subtract the exponents from one another.
Example: $3^3 / 3^2 = (3 \times 3 \times 3) / (3 \times 3) = 3^{(3-2)} = 3^1$

By this method then we can find the value of a number with the exponent 0.
Example: $1 = (3 + 3 + 3) / (3 + 3 + 3) = (3 \times 3) / (3 \times 3) = 3^2 / 3^2 = 3^{(2-2)} = 3^0 = 1$

~~~~~

<u>Roots</u>

Finding a natural number root of a natural number is a much more complicated and specific type of *division*.
~~~~~

To find a natural number root of a natural number is to find a factor (divisor) of the natural number that divides evenly and completely into the natural number *one or two* or more times.

We divide the factor of a natural number into the natural number until we are left with only the number 1.

When dividing (or multiplying) 1 isn't a "remainder". It is known as the "multiplicative identity". Any number multiplied or divided by 1 is the same number: (a x 1 = a), (a / 1 = a)

A natural number of course can always divide into itself *one* time, and so a natural number is always the *first* root of itself. (For the most part we will only look at roots of a number that are greater than *one*.)

Very few natural numbers have natural number roots, (other than the *first* root).

For example, the factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18 and 36. Of all the possible factors only 6 divides *twice* into 36 evenly (36 / 6 / 6 = 1). As 6 divides *twice* into 36 it is called the *square root* of 36 ($\sqrt{36}$). Inversely 6 squared (6^2) equals 36.

We can also use prime factorization of a natural number to determine if it has any natural number roots.

For example, the prime factors of 36 are (2 x 2 x 3 x 3). We can with these prime factors make two similar pairs of (2 x 3) and (2 x 3), which of course is (6 x 6) or (6^2), which equals our original number of 36. 6 then is the *square root* of 36 as it divides evenly into 36 *twice*.

For a second example, the factors of 64 are 1, 2, 4, 8, 16, 32 and 64.
The natural number 64 has several different natural roots.

2 divides evenly 6 times into 64 (64 /2/2/2/2/2/2 = 1),
so 2 is the *sixth root* of 64, written $\sqrt[6]{64} = 2$

4 divides evenly 3 times into 64 (64 /4/4/4 = 1),
so 4 is the *third* or *cube root* of 64, written $\sqrt[3]{64} = 4$

8 divides evenly 2 times into 64 (64 /8/8 = 1),
so 8 is the *second* or *square root* of 64, written $\sqrt[2]{64} = 8$

The prime factors of 64 are 2, 2, 2, 2, 2, 2 which can be equally arranged as follows:

(2) x (2) x (2) x (2) x (2) x (2) which equals 2^6, so inversely 2 is the *sixth root* of 64.
(2 x 2) x (2 x 2) x (2 x 2) which equals 4^3, so inversely 4 is the *cube root* of 64.
(2 x 2 x 2) x (2 x 2 x 2) which equals 8^2, so inversely 8 is the *square root* of 64.

For a third example, the factors of 48 are 1, 2, 4, 6, 8, 12, 16, 24 and 48.
Even though 48 has many factors no factor divides evenly into 48 *two* or more times.

We can confirm this by finding the prime factors of 48 which are 2, 2, 2, 2, 3. It is obviously not possible to arrange these factors into *two* or more equal groups.

When seeking then the natural number root of a natural number we are looking for a factor of the natural number that can divide evenly and completely into the natural number *(one or) two* or more times until we are left with the number 1. The number of times the factor can divide into the original natural number determines the degree of the root.

Example: 243 / 3 = 81, 81 / 3 = 27, 27 / 3 = 9, 9 / 3 = 3, 3 / 3 = 1
The *fifth root* of 243 then is 3, as 243 = 3 x 3 x 3 x 3 x 3 = 3^5

A **square root** of a number evenly divides into that number *two* times. It is called a square root as we are starting with the *area* of a square and are looking for the length of its two equal and perpendicular sides. Not all arbitrarily chosen areas give natural number sides. This will be endlessly problematic.
The symbol for a square root is: √
Example: √25 = 5 (as inversely 5 x 5 = 5^2 = 25)

A **cube root** of a number evenly divides into that number *three* times. It is called a cube root as we are starting with the *volume* of a cube and looking for the length of its three equal and perpendicular sides. Not all arbitrarily chosen volumes give natural number sides. This will be endlessly problematic.
The symbol for a cube root is: ^{3}√
Example: ^{3}√216 = 6 (as inversely 6 x 6 x 6 = 6^3 = 216)

We can continue searching for higher number roots of a number: ^{x}√
Example: ^{6}√729 = 3 (as 3 x 3 x 3 x 3 x 3 x 3 = 3^6 = 729)

Of course, as we mentioned, we can also have the *first* root of any number, which is also just the same as the number to the exponent 1, which finally is just the same as the number itself. Example: $\sqrt[1]{7} = 7^1 = 7$

~~~~~

Roots can also be written as fractional exponents: ½ ⅓ ¼ and so on.
(We will look at rational numbers further on).

The square root of 9 which equals 3, can be written as $9^{1/2}$ (as $9^{1/2} \times 9^{1/2} = 9^1$)
The cube root of 64 which equals 4, can be written as $64^{1/3}$ (as $64^{1/3} \times 64^{1/3} \times 64^{1/3} = 64^1$)
The quad root of 525 which equals 5, can be written as $525^{1/4}$
(as $525^{1/4} \times 525^{1/4} \times 525^{1/4} \times 525^{1/4} = 525^1$)
And so on.

This works, for when we multiply together same numbers that both have exponents (that is numbers having the same base), we just add the exponents.
Example: $3 \times 3 = 9^{1/2} \times 9^{1/2} = 9^1 = 9$

Another way of writing exponents and roots on a single line is with the symbol **^**.
Example: 9**^**2, 16**^**(1/2) and so on.

~~~~~

Summary of Exponents and Roots

Exponents *multiply* the same number by itself one or many times seeking a total value or product. A natural number to a natural number exponent will always be a natural number.

To find the **Root** of a natural number is to seek a factor (divisor) of that number that *divides* evenly and completely into the number (*one or) two* or more times until we are left with the "multiplicative identity" of 1. Very few natural numbers have natural number roots greater than *one*.

~~~~~

Summary of Mathematical Operators

Of greatest interest is how, when we move from simple addition and subtraction, to multiplication and division, and then to exponents and roots, we are always just dealing with the manipulation of a *single number repeated many times*!

Multiplication is just adding the same number together many times finding a sum.
Example: 3 x 5 = 3 + 3 + 3 + 3 + 3 = 5 x 3 = 5 + 5 + 5 = 15
(But we rarely think of 15 = 1 + 2 + 3 + 4 + 5)

Division is just finding the size of an equal part after a number has been divided into one or more equal parts.
Example: 15 / 5 = 3 as 3 + 3 + 3 + 3 + 3 = 3 x 5 = 15

We can also think of division as from a *total* subtracting an *equal part* a number of times.
Example: 15 / 5 = 3 as 15 -3 -3 -3 -3 - 3 = 0

Exponents are simply multiplying the same number together one or more times, (which can also be expanded into addition).
Example: 3^3 = (3 x 3) x 3 = (3 + 3 + 3) + (3 + 3 + 3) + (3 + 3 + 3) = 27

Roots are finding a factor (divisor) of a number that divides equally into that number (*one or) two* or more times, (which can also be expanded into subtraction)
Example: $\sqrt[4]{16}$ = 2 as 16 = 2 x 2 x 2 x 2
and so 16 = (2 x 2) x (2 x 2) = (2 + 2) + (2 + 2) + (2 + 2) + (2 + 2)
or 16 - (2 x 2) x (2 x 2) = 16 -2 -2 -2 -2 -2 -2 -2 -2 = 0

Multiplication, division, exponents, and roots can all be reduced to simple addition (or subtraction), and not *any* type of simple addition (or subtraction), *but* the manipulation of the *repetition of a single number value*!
~~~~~

Properties of Natural Numbers

A number of fundamental properties have been formulated for **natural numbers**, many of which seem very simple, obvious and self-explanatory.

~Closure
For natural numbers a and b, the sum of $a + b$, and the product of $a \times b$, are also both natural numbers.
Example: $4 + 5 = 9$ and $4 \times 5 = 20$

For natural numbers this does not hold for *all* subtraction and *all* division.
Example: $4 - 5 \neq$ natural number, $4 / 5 \neq$ natural number

~Associativity
For natural numbers a, b and c, $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$
Example: $4 + (5 + 6) = (4 + 5) + 6 = 15$
$4 \times (5 \times 6) = (4 \times 5) \times 6 = 120$

~Commutativity
For natural numbers a and b, $a + b = b + a$ and $a \times b = b \times a$
Example: $4 + 5 = 5 + 4$ and $4 \times 5 = 5 \times 4$

~Distributivity
For natural numbers a, b and c, $a \times (b + c) = (a \times b) + (a \times c)$
Example: $3 \times (4 + 5) = (3 \times 4) + (3 \times 5) = 27$

~Identity Elements
The “additive identity” element is 0 as $(a + 0)$ or $(a - 0) = a$
The “multiplicative identity” element is 1 as $(a \times 1)$ or $(a / 1) = a$

~~~~~
**~Unique or Prime Factorization Theorem**
Every natural number can be uniquely factorized into Prime numbers in only one way, which in a way is its unique “genetic” identity.
Example: $17 = 17$, $28 = 2 \times 2 \times 7 = 2^2 \times 7$, $54 = 2 \times 3 \times 3 \times 3 = 2 \times 3^3$

This is also called the **Fundamental Theorem of Arithmetic**
~~~~~

The factors of a number are just all the different ways the prime factors of that number can be multiplied together. As the number 1 is debatably excluded from being a prime number, we must manually add it to our list of factors of a number.

Example: The prime factor of 17 is 17
The factors of 17 are 1 ,17

The prime factors of 28 are 2 x 2 x 7
The factors of 28 are 1, 2, 4, 7, 14, 28

The prime factors of 54 are 2 x 3 x 3 x 3
The factors of 54 are 1, 2, 3, 6, 9, 18, 27, 54

~~~~~

Qualities of Natural Numbers

A **natural number** can be either **0**, the unit **1**, **composite** or **prime** (non-composite).

0 ~Much has been written about the number **0**, which we won't go into here. Suffice it say, "*after we eat one apple we will have zero apples left",* so it doesn't make very much sense to exclude **0** from being a natural number!

1 ~Some ancient Greeks felt that **1** is different from all other numbers, or not even a number, as being the most fundamental *unit*, its multiplicity is the basis of all other numbers.

I never find the argument for why the number **1** isn't a **prime number** very satisfying.
1 obviously isn't a **composite natural** number, and if it isn't a **prime number** then what is it?
1 then is left to be the *unit* that is unique unto itself, as philosophically in times past it has been.

~A **composite natural number** is the product of two (or more) natural numbers multiplied together (*excluding 0 and 1*). This means that a **composite natural number** must be a number (*greater than 1*) added to itself more than one *time*.
Example: 10 = 2 + 2 + 2 + 2 + 2 = 2 x 5 or 10 = 5 + 5 = 5 x 2

~A **prime natural number** is a number that can't be made up by adding a number (*greater than 1*) to itself more than one *time*. This means that it is a number whose factors are only itself and the number 1. All prime numbers are natural numbers.
Example: 17 = 1 x 17
~~~~~

A simple proof tells us that there are an “endless number of prime numbers”:

~Take the product of any number of primes (in order, starting with 2) and add 1.
Example: (2 x 3 x 5 x 7) + 1 = 210 + 1 = 211

~The new number obviously cannot be divisible by any of the primes we started with.
In the example, the number 211 divided by 2, 3, 5 or 7, would all have the remainder of 1.

~Therefore either, 1) the new number must be a new prime number, or
2) it is divisible by a prime factor greater than our initial prime numbers

Both of our conclusions would imply a Prime number or Prime factor greater than the Prime numbers we started with!

The **Unique or Prime Factorization Theorem**, also called the **Fundamental Theorem of Arithmetic**, is the main reason the number **1** isn’t considered a Prime Number. If **1** was a Prime number, then **1** would be a Prime factor of *every* natural number, which is deemed to be unnecessary or even redundant.

The number **1**, (whether considered to be a Prime number or not), is not necessary for our proof that there are an “endless number of Prime numbers”.

But, when above, we use the Prime factors of a number to determine <u>*all*</u> possible factors of that number, we must manually reintroduce the number **1** as a factor of that number. As **1** isn’t considered a Prime number, it can’t be used as a Prime factor to determine all the possible factors of a number. But the factors of *all* natural numbers must always include the number **1.**

And so, we can see that the number **1** sometimes is necessary and logical as a Prime Number, and at other times not! So, it might be better to say that the number **1** is an **occasional prime** depending on the context it is used in!

To say that **1** can’t be a prime number because it only has one factor (itself), and that all prime numbers need to have two *distinct* factors, (the prime number itself and the factor 1), is an entirely arbitrary and groundless rule or condition.

~~~~~
Natural numbers can also be divided into **even** numbers and **odd** numbers.
All **even** numbers can be divided by 2.
Example: 2, 4, 6, 8,…

The numbers that can’t be divided into 2 are called **odd** numbers.
~~~~~

Sometimes **0** is thought of as an even number but mostly it is unique to itself.

There is no reason why we can't as well make up unique sets of numbers that can consist of numbers that can be divided by any natural number.

Example: Natural numbers divisible by 3: 3, 6, 9, 12,…
Natural numbers divisible by 4: 4, 8, 12, 16,…
And so on.

Mathematical Induction

~**"Mathematics"** (noun) may be simply defined as, "*the science or study of numbers, quantity, space, etc., either abstractly or practically*". The word "**mathematical**" is an adjective meaning, *"of or relating to **mathematics**".*

~The word "**Induction"** can be considered an "action noun". "Action nouns" are derived from closely related verbs, and describe or express an "*action, process, or state of being".* Some examples of "action nouns" include *"belief, movement, existence, creation",* which are derived from the verbs "*believe, move, exist, create"* respectively. An "action noun" represents a verb's meaning as a noun, and basically converts an "action" (verb) into an "entity" (noun).

The word "**induction**" (action noun) is derived from the verb "**induce**". The word "**induce**", pertaining to *logic,* means *"to derive or bring about by **inductive** reasoning".* It is not common to use the verb "**induct**" in a *mathematical* or *logica*l context. It is pretty much impossible to find a definition of the verb "**induct**", as pertaining to *mathematics* or *logic*, in any dictionary!

"***Inductive reasoning***" starts with specific observations to draw *general* conclusions. "***Deductive reasoning***" on the other hand, starts with general principals to derive *specific* conclusions.

The word "**inductive**" (adjective), as pertaining to *logic*, means "*a type of **inference** where general laws are deduced from particular instances",* or as above, "*a kind of reasoning where general conclusions are drawn from specific observations or evidence*".

The word "**inference**" (action noun) means "*the process of reaching a conclusion based on reasoning and evidence*".

The verb "**infer**" is derived from the Latin "***inferre***", which means to "*to carry in, bring into, conclude, deduce*".

The words "***induction***", "***induct***", "***induce***", and "***inductive***", are all derived from the Latin verb "***inducere***". The Latin prefix "**in-**" in this case means: "*in, within, into, on, upon"*. The Latin verb "***ducere***" means: "*to lead, guide, command, prolong, think, consider, regard*".

Putting the two together for "***inducere***" we get: "*to lead in*", "*to guide within*", "*to think upon*", and so on, depending on context.

It is obviously quite complicated to define the word **Induction** as pertaining to *mathematics* and *logic*!

~~~~~

The goal of Mathematical Induction is to prove that a statement, formula, or theorem is true for all **natural numbers** n, (usually where n ≥ 0 or n ≥ 1), without having to prove the statement, formula, or theorem, true for each **natural number** individually, which of course is impossible!

Our statement, formula or theorem can also be called our predicate, property, or proposition, and its formulation is called the *propositional function* P(n). P(n) is not a mathematical function but a *logical* assertion that refers to the **natural number** n.

Mathematical Induction is mainly used with **natural numbers**, as simply, one natural number always follows another natural number, and so on forever.

A proof by Mathematical Induction is carefully put together in three steps, with a beginning, middle and end, and the proof should be clear, concise, logical and pleasing. A proof by Mathematical Induction should be as close as possible to a demonstration of pure logic, which is interesting in itself.

~~~~~

Mathematical Induction has three steps. It is always difficult to explain concepts right off the bat with just words, so an example will follow as well.

The ***first step***, called the **base step** (or **base case**), is to ***show*** that our propositional function P(n) is *true* for either P(0) or P(1), meaning either the **natural number** n = 0 or n = 1.

Sometimes a higher value of n is used to show that the proposition P(n) is *true* in the **base step**. This first step must be *true* otherwise there is no going on.

The ***second step***, called the **induction hypothesis**, is where we ***assume*** that the proposition P(k) is true for (*any* number n). While n is always defined as (*some* number n), k stands for (*any* number n).

We know by our **first** or **base step** that P(k) is *true* when k equals the n of our first step, (usually either n = 0 or n = 1). We will ***assume*** that P(k) is also true when k = (*any* n), (where n is equal to or greater than the n of our first step, which is usually either n = 0 or n = 1).

For **Weak Induction** we ***assume*** only a single step of P(k) equaling (*any* number n) is true.
For **Strong Induction** we ***assume*** a number of steps of P(k) equaling (*any* number n) are true.

Weak (or Simple) Induction is the regular type of induction.
Strong Induction means ***assuming*** a number of steps are *true* before proceeding to the third step of P(k + 1).

The ***third step***, called the **inductive step** is where we ***prove*** that the proposition P(k+1) is true in relation to P(k), or in other words, P(*any* n + 1) is true in relation to P(*any* n). As we know our ***first step*** is true for P(n), then all subsequent values of n must be true as well, as we have shown that all P(n+1) are true in relation to all P(n), (where n is equal to or greater than the n of our first step, usually either n = 0 or n = 1).

There are then three major steps to mathematical induction that we can also think of as ***show, assume*** and ***prove.***

In our first step we **show** that the proposition P(n) is true for (usually) n=0 or n=1.
In our second step we **assume** that the proposition P(k) is true for k equal to (*any* number n) (where n is equal to or greater than the n of our first step).
In our third step we **prove** that the proposition P(k+1) is true in relation to P(k), where (k+1) is equal to (*any* number n+1) and k is equal to (*any* number n).

As k is equal to (*any* number n), then k must also be equal to the number n of our first step (usually n=0 or n=1), whose proposition P(n) we know to be true.

As P(k+1) is true in relation to P(k), so must P(*any* number n+1) be true in relation to P(*any* number n).

And so, we may then ***mathematically induct*** (adjective + verb) that the proposition P(n) is true for all natural numbers n, (where n is equal to or greater than the n of our first step).

~~~~~

The following **example** will provide more depth, detail, and clarity on ***Mathematical Induction***.

Example:

We set out to prove the proposition P(n) that the "sum of all odd natural numbers always gives us a square number". (n) will also tell us the number of terms we have on the left side of our proposition.

P(n): $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

For example, the first four terms (n = 4) give us $1 + 3 + 5 + 7 = 16 = 4^2$

We can try for fun a few other values for n.

(n cannot equal 0 as there would then be *no* terms on the left side of our proposition! *Nothing* (no terms) cannot equal *something* (0^2).)

n = 1 gives $1 = 1^2$

n = 2 gives $1 + 3 = 2^2$

n = 3 gives $1 + 3 + 5 = 3^2$

n = 5 gives $1 + 3 + 5 + 7 + 9 = 5^2$

We might think that if our proposition works for a number of different values of n then it should work for all n. Sometimes just checking a few possible values of n gives us a false confidence that the formula works for all n. But that isn't good enough mathematically. We still need to rigorously prove by ***Mathematical Induction*** that our proposition works for *all* n from our base step up.

Our ***first step*** (the ***base step***) is to ***show*** that our proposition P(n) holds for either P(0) or P(1).

For P(0), where n = 0, we have no terms at all in our sequence of odd numbers on the left hand side of the equation, (and so, the proposition P(n): $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ is not even valid when n = 0).

For P(1), where n = 1, then our proposition P(n) is true,

as $(2n - 1) = n^2$ is true, giving $1 = 1^2$, which is also the first odd number in our sequence.

We will be trying then to prove our proposition for all natural numbers n where $n \geq 1$.

If our base step proposition is false for every value of n that we might try, then there is no going on.
~~~~~

In our ***second step*** (the ***induction hypothesis***), we ***assume*** that our proposition P(k) is true for k = (*any* n), where n ≥ 1.

P(k): 1 + 3 + 5 +…+ (2k-1) = k^2, where k = (*any* n), (n ≥ 1)

This is a very straight forward step where all we have done is replace n in our proposition with k. We are saying that k = (*any* number n), where n ≥ 1. We are ***assuming*** that our proposition P(k) is *true* for k = (*any* n), (n ≥ 1), without checking to see if it is actually true, as it is impossible to check every n anyways! We know at least that when k equals the n of our ***first step***, our proposition P(k) is true.

In our ***third step*** (the ***inductive step***), we try to ***prove*** that the proposition P(k+1) is true only in *relation* to P(k), where (k+1) = (*any* number n+1) and (k) = (*any* number n), (n ≥ 1)

The ***third step*** is where we have to be clever and prove in some way our original proposition P(n) for all n. This method of proof will differ for different propositions.

~~~~~

The following is the method we will use for the ***third step*** of this proposition:

The next odd term following [1 + 3 + 5 +…+ (2k-1)] is (2k+1).

If for example, k = 5 then the first five terms of proposition P(k) will be [1 + 3 + 5 + 7 + 9]

The sixth term that follows will be 11 which equals (2k+1).

If for example, k = 5 and we have 6 terms, our new proposition then must become P(k+1).

Our new proposition P(k+1) would then be written:

[1 + 3 + 5 +…+ (2k-1)] + (2k+1) = $(k+1)^2$

We need to show that our new proposition P(k+1) is *true* in relation to P(k).

From our ***second step*** we *assume* that:

[1 + 3 + 5 +…+ (2k-1)] = k^2
~~~~~

So, we can for our new proposition P(k+1) substitute k^2 in for [1 + 3 + 5 +…+ (2k-1)],

and so,

$k^2 + (2k+1) = (k+1)^2$

which is true as $(k+1)^2$ expands to $k^2 + 2k + 1$.

Therefore, we have shown that at least the new proposition P(k+1) of the ***third step*** is *true* in relation to the proposition P(k) of the ***second step***.

We know that our ***first step*** is true, and our ***third step*** is true in relation to the ***second step***,

We also know the ***second step*** is true in relation to the ***first step***.
From the proposition P(k) of the ***second step***, where k equals (*any* number n) (n ≥ 1), k must also equal the number n of our **first step**, which we know is true.

And so, if our ***third step*** is true in relation to the ***second step,*** and our ***second step*** is true in relation to the ***first step***, then our ***third step*** must also be true in relation to the ***first step***!

And so, like a long endless chain, every subsequent P(n+1) is true in relation to the previous P(n), and we know that the first P(n) we started with is true.

And so, we have proven our proposition P(n) for all n ≥ 1.

Therefore P(n): $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ is true for all n ≥ 1.

~~~~~

We have to wrap our head around the subtlety of this proof which may leave us asking a few questions about its method and validity.

"***Inductive reasoning***" starts with *specific* observations to draw *general* conclusions. The question is, how can we know for sure that we can extend the *truth* of our *specific* observations to the *truth* of a *general* conclusion? Usually, a middle term needs to be inserted between the *specific* observation and the *general* conclusion to connect the two premises *truthfully.*
~~~~~

There are endless *logical and philosophical* arguments as to what strength or validity (*strong, weak, cogent, not cogent*) an argument by "***inductive reasoning***" can actually be known to be certainly true*.* The adjective "cogent" is simply defined as "*clear, logical, convincing",* and is usually used in the context of an *argument, reason or example;* for example, a "*cogent argument",* a "*cogent reason"* or a "*cogent example*".

A proof by ***Mathematical Induction***, if done correctly, must be as true as any other acceptable mathematical proof.

~~~~~

Mathematical Induction Summary

In the first step we ***show*** the truth of the proposition P(n) for a fixed natural number, usually n = 0 or 1, though n can also be any other natural number.

In the second step we ***assume*** that the proposition P(k) is true when k equals (*any* number n). We know at least that our proposition P(k) is true when k equals the n of our first step.

In the third step we set out to ***prove*** that the proposition P(k+1) is true only in relation to P(k), where (k+1) equals (*any* number n+1) and k equals (*any* number n).

And so, we prove our proposition P(k+1) for (*any* n+1) is true related to P(k) for (*any* n).

But since k equals (*any* number n), k must also equal the number n of our first step, whose proposition P(n) we have already shown to be true.

And so, every subsequent P(n+1) must be true in relation to every previous P(n).

Therefore, by showing that our three steps are all true in relation to one another, we prove that the proposition P(n) is true for *all* n, (where n is equal to or greater than the value of n in our first step).

And so, we finish our proof by ***Mathematical Induction***.
~~~~~

Conclusion

Natural Numbers can increase in magnitude endlessly, but *individually* never cease to be **finite**.

Only "*as a whole*" can **Natural numbers** be said to be **infinite**. It is not possible to *"measure or define"* the largest **Natural number**, as they are without end.

Adding and multiplying **Natural numbers** always gives another **Natural number**.
Subtraction and division using **Naturals numbers** is limited to positive whole numbers.

All multiplication and exponents using **Naturals numbers** (and inversely division and roots) can be reduced to a single number value repeatedly added to itself, or repeatedly subtracted from a total (with or without a remainder).

There are many varied and subtle ways of proving Propositions by **Mathematical Induction**.
Almost all of them use **Natural numbers**.

Things become very interesting on so many levels when we interface or connect the logic of our mind with **Natural numbers** and start asking interesting questions!

The study of the properties of **Natural numbers** (along with **Integers**) is called **Number Theory**, which in a simple way is the search for **Natural number** patterns.

Chapter 2

Integers [Z] …$^{-}$5, $^{-}$4, $^{-}$3, $^{-}$2, $^{-}$1, 0, $^{+}$1, $^{+}$2, $^{+}$3, $^{+}$4, $^{+}$5…

We can use the negative and positive **Finite** (or **Adfinite**) symbols $^{-}$∝ and $^{+}$∝ (open lemniscates) to represent the negative and positive **Integers** that endlessly decrease and increase forever without ever ceasing to be **finite**.

We can write then:
Integers [**Z**] = $^{-}$∝…$^{-}$5, $^{-}$4, $^{-}$3, $^{-}$2, $^{-}$1, 0, $^{+}$1, $^{+}$2, $^{+}$3, $^{+}$4, $^{+}$5…$^{+}$∝

All Integers are **Finite**.

The symbol **Z** is used to symbolize the set of Integers.

Integers consist of the **Natural numbers** including **zero**, and the numbers that are the ***negative*** opposite of the ***positive*** **Natural numbers**.

We have looked extensively at the **Natural numbers** including **zero**. We now have to learn how to understand, define, and think about ***negative*** **Integers**, both conceptually and mathematically.

~~~~~

How can we think of **negative numbers**?

Natural numbers are simply defined as “counting numbers” that tell us the quantity, or how many of something we have. **Negative numbers** being in a way “opposite” to the **positive Natural numbers**, tell us how many of something we *don’t* have. We can and do interpret this as “owing” something, being in “debt”, or having a “deficit”.

While **negative numbers** have been around and accepted for thousands of years, some notable figures like *Diophantus* and *Leibniz* considered them to be false, absurd, or invalid.

So, the concept of **negative numbers** corresponds quite easily with our sense of reality!
If we have something, we are in the **positive**.
If we don’t have something, we have **zero**.
If we owe something, we are in the **negative**.
~~~~~

Positive and Negative symbols

To help conceptualize **Integers** we can use what is called a **Number Line**. In the center of the number line is the number **zero** which is neither positive nor negative. To the right of zero are all the **positive integers** stretching out forever to **positive adfinity**. To the left of zero are all the **negative integers** stretching out forever to **negative adfinity**.

Integers can be both added to and subtracted from forever without end. No matter how great or small they get as they increase (positive) or decrease (negative), *individual* **integers** never cease to be **finite**. At no point do **integers** cease to be "*countable*", or magically at some mystical point become **in-finite**, meaning *"not"* **finite**, or simply, not "*measurable or definable*".

Integers share the same **finite** / **infinite** duality as **natural numbers** do.

All **Integer** are *"countable"*. Any *individual* **integer** we choose automatically has a "*bound, limit*, *size* or *extent"* and is "*measurable or definable"*.

Integers however are *"countable"* forever in *both* the positive and negative direction. Only *as a whole* can **integers** be said to be without "*bounds, limit*, *size* or *extent".* It is not possible to *"measure or define"* the greatest positive, or the smallest or least negative **integer**, as they are without end. Only *as a whole* can **integers** be said to be **in-finite**, meaning without "*bounds, limit*, *size, extent"*, and without a *"measurable or definable limit".*

~~~~~

With **Natural numbers** we took for granted that any whole number or "counting number" was always positive, or zero. Of course, this isn't the case for **Integers**. For example, we can have now both -5 and +5. Convention is to ambiguously write, for example, -5 and 5, leaving off the positive symbol as we did with Natural numbers.

But…with **Integers**, the positive and negative symbols are no longer just *mathematical operators*. Conventionally, the *same* symbols are used as *mathematical operators*, *and* to tell us whether a number *value* is positive or negative. But…a positive or negative symbol that expresses the *value* of an **Integer** has *nothing* to do with the same symbol as a *mathematical operator*!

In the chapter on **Natural numbers** we used the positive and negative symbols only as *mathematical operators,* and just took for granted that every natural number was positive, or zero.

So then, to distinguish a *mathematical operator* from a *value,* we can write the positive and negative symbols for *values* in superscript. Our number line of *values,* stretching out either way to positive and negative **adfinity**, would then look like this:
~~~~~

$\underline{{}^{-}\propto \ldots\ {}^{-}5\ \ {}^{-}4\ \ {}^{-}3\ \ {}^{-}2\ \ {}^{-}1\ \ 0\ \ {}^{+}1\ \ {}^{+}2\ \ {}^{+}3\ \ {}^{+}4\ \ {}^{+}5 \ldots {}^{+}\propto}$

It is currently common practice, when showing the positive or negative *value* of a number, to put the number *value* into brackets, for example (-5) **+** (+3).

For **Integers**, using superscripted negative and positive symbols for *values*, and regular positive and negative symbols for *mathematical operators*, we need to explore a number of possible combinations, which we will do in the next section.

Mathematical Operators

We will look again at six mathematical operations in three related pairs, but for **Integers**:
~ Addition and Subtraction
~ Multiplication and Division
~ Exponents and Roots

~~~~~
### Addition and Subtraction

We will start with any **integer** value on the number line and designate it as **a**.

To **a** we will add a superscript to show that **a** can be either negative, zero or positive: $^{-/0/+}\mathbf{a}$

To this value we will add or subtract a second **integer**: $^{-/0/+}\mathbf{b}$

We will determine if the resultant sum on the number line moves to the **right** (increases), moves to the **left** (decreases), or stays the same as the initial value which is $^{-/0/+}\mathbf{a}$.

There are six possibilities to try.

The first two possibilities include adding and subtracting zero from $^{-/0/+}\mathbf{a}$.
Zero is called the “additive identity”.

1) $( ^{-/0/+}\mathbf{a} + {}^{0}\mathbf{b} ) = {}^{-/0/+}\mathbf{a}$.
2) $( ^{-/0/+}\mathbf{a} - {}^{0}\mathbf{b} ) = {}^{-/0/+}\mathbf{a}$.

Our initial $^{-/0/+}\mathbf{a}$ value remains unchanged when zero ( $^{0}\mathbf{b}$ ) is added to or subtracted from it.

We have four other possibilities where **b** doesn’t equal zero.
We will write each of these four possibilities out in three different ways:

3) $^{-/0/+}\mathbf{a} + {}^{+}\mathbf{b}$ **= right** ***or*** **a + (+b) = right** ***or*** **a + b = right**

4) $^{-/0/+}\mathbf{a} + {}^{-}\mathbf{b}$ **= left** ***or*** **a + (-b) = left** ***or*** **a - b = left**

This so far makes straight forward sense.
~~~~~

If we ***add*** a positive *value,* we increase the value we started with, and so move to the **right** on the number line.

If we ***add*** a negative *value,* we are increasing our “debt” or “deficit”, and so we can only move to the **left** on our number line.

We can show in a simple example how negative numbers can be thought of as the opposite of positive numbers.

a) (+**3**) **+** (+**4**) **=** (+**7**)   b) (-**3**) **+** (-**4**) **=** (-**7**)

Things can get a little more tricky subtracting values using the negative operator.

5) $^{-/0/+}$**a - $^{+}$b = left** ***<u>or</u>*** **a -** (+**b**) **= left** ***<u>or</u>*** **a - b = left**

6) $^{-/0/+}$**a - $^{-}$b = right** ***<u>or</u>*** **a -** (-**b**) **= right** ***<u>or</u>*** **a - -b = right**

We can see that we can also make straight forward sense here.

If we ***subtract*** a positive value, we decrease the value we started with, and so move to the **left** on the number line.

If we ***subtract*** a negative value, we are decreasing our “debt” or “deficit”, and so we can only move to the **right** on our number line.

We are entering the confusing world where two negatives equal a positive, but we can see how easy this can be to conceptualize in the real world and not abstractly, especially when separating positive and negative *values* from positive and negative *mathematical operators*.

We can see why positive and negative symbols conventionally do double duty for being both a *value* and a *mathematical operator*. Only when subtracting a negative value (**a - -b**) do we have to show both negative signs. Both (**a -** (+**b**)) and (**a +** (-**b**)) are simply written as (**a - b**), as they give the same numerical result, even though they aren’t the same thing!

~~~~~

N**atural numbers** are considered to be “closed” only for addition and multiplication, as adding or multiplying any natural numbers together gives another natural number. For **integers**,
~~~~~

subtraction is also considered to be "closed", as adding or subtracting any integers together gives another integer. Division only becomes "closed" when we start using **rational numbers**.

~~~~~

Multiplication

Again, multiplication is just a form of addition where the same number is added to itself a number of times. But how does this work for **integers**, since we also have negative values to deal with?!

As with **natural numbers**, the *value* we are multiplying is called the **multiplicand**.
The number of times the multiplicand is added to or subtracted from itself is shown by the **multiplier**. The answer or result in multiplication is called the **product**.

multiplicand x multiplier = product

It is very easy to extend our definition of the **multiplicand** and **multiplier** to include negative numbers.

Our **multiplicand** and **multiplier** can both be ***positive*** and **zero**, and now also ***negative***, giving us nine possibilities:

	Multiplicand	**Multiplier**
1)	Positive	Positive
2)	Negative	Positive
3)	Zero	Positive
4)	Positive	Negative
5)	Negative	Negative
6)	Zero	Negative
7)	Positive	Zero
8)	Negative	Zero
9)	Zero	Zero

We will look at each possible combination. We will show first the ***multiplicand*** in expanded form, with both *mathematical operators, and* positive or negative superscripts or zeros for the *value* of each ***multiplicand.*** We will then show how this is reduced to "***multiplicand*** *times* ***operator*** *equals* ***product***", again always showing both *values* and *mathematical operators.*
~~~~~

~~~~~

$^{+/-/0}$Multiplicand x $^{+}$Multiplier = $^{+/-/0}$Product

A ***positive*** **multiplier**, for positive, negative and zero **multiplicands**, then is a short form for how many times we have repeated the *mathematical operation* of ***addition***, (symbolized by the "**+**" *mathematical operator*), to add the **multiplicand** to itself.

In full, using *positive/negative* symbols for both *values* and *mathematical operators* we could write:

1) For positive *value* **multiplicand** integers and a positive **multiplier**:

Example: **+** $^{+}$5 **+** $^{+}$5 **+** $^{+}$5 **+** $^{+}$5 = $^{+}$5 **x (+**$^{+}$4) (meaning $^{+}$5 **added** together $^{+}$4 *times*) = $^{+}$20

+ $^{+}$4 **+** $^{+}$4 **+** $^{+}$4 **+** $^{+}$4 **+** $^{+}$4 = $^{+}$4 **x (+**$^{+}$5) (meaning $^{+}$4 **added** together $^{+}$5 *times*) = $^{+}$20

Normally written: 5 **x** 4 = 20 and 4 **x** 5 = 20

2) For negative *value* **multiplicand** integers and a positive **multiplier**:

Example: **+** $^{-}$5 **+** $^{-}$5 **+** $^{-}$5 **+** $^{-}$5 = $^{-}$5 **x (+**$^{+}$4) (meaning $^{-}$5 **added** together $^{+}$4 *times*) = $^{-}$20

+ $^{-}$4 **+** $^{-}$4 **+** $^{-}$4 **+** $^{-}$4 **+** $^{-}$4 = $^{-}$4 **x (+**$^{+}$5) (meaning $^{-}$4 **added** together $^{+}$5 *times*) = $^{-}$20

Normally written: $^{-}$5 **x** 4 = $^{-}$20 and $^{-}$4 **x** 5 = $^{-}$20

Let us continue on with a zero *value* for the ***multiplicand***.

As a *value* zero could then be written ^{0}0 as it is neither positive nor negative. But for simplicity's sake we could also just write the *value* zero as 0, as the superscript for zero serves no extra defining purpose, (as do the positive and negative superscripts for the *values* of the positive and negative integers).

We saw earlier, for example, that 0 x 7 = 0 + 0 + 0 + 0 + 0 + 0 + 0 which equals 0.

We can write this now with our more complicated, but comprehensive notation:

3) For a zero *value* **multiplicand** integer and a positive **multiplier**:

Example:

+ 0 **+** 0 **+** 0 **+** 0 **+** 0 **+** 0 **+** 0 = 0 **x (+**$^{+}$7) (meaning 0 **added** together $^{+}$7 *times*) = 0

Normally written: 0 **x** 7 = 0
~~~~~

We can see for the **operator** (in brackets), how we use both a positive *mathematical operator,* and a positive *value* for how many *times* we **add** the positive, negative, or zero *value* of the **multiplicand** together.

~~~~~

But, what if our **multiplier** is negative? What can that possibly mean?

We can easily understand for example, what ($^{+}5$ **x** $^{+}4$) or ($^{+}5$ added together $^{+}4$ *times*) means. But what does it mean when we multiply our ***multiplicand*** with a negative ***multiplier***, for example: ($^{+}5$ **x** $^{-}4$)?

Maybe it means that our negative ***multiplier*** is a short form for how many *times* we use the **negative** *mathematical operator* to ***subtract*** the ***multiplicand*** from itself?

$^{+/-/0}$**Multiplicand x $^{-}$Multiplier = $^{-/+/0}$Product**

A ***negative*** **multiplier**, for positive, negative and zero **multiplicands**, then is a short form for how many times we have repeated the *mathematical operation* of ***subtraction***, (symbolized by the "**-**" *mathematical operator*), to subtract the **multiplicand** from itself.

Let us try it!

4) For positive *value* **multiplicand** integers and a negative **multiplier**:

Example: **-** $^{+}5$ **-** $^{+}5$ **-** $^{+}5$ **-** $^{+}5$ = $^{+}5$ **x (-**$^{+}4$) (meaning $^{+}5$ **subtracted** together $^{+}4$ *times*) = $^{-}20$
- $^{+}4$ **-** $^{+}4$ **-** $^{+}4$ **-** $^{+}4$ **-** $^{+}4$ = $^{+}4$ **x (-**$^{+}5$) (meaning $^{+}4$ **subtracted** together $^{+}5$ *times*) = $^{-}20$

Normally written: 5 **x** $^{-}4$ = $^{-}20$ and 4 **x** $^{-}5$ = $^{-}20$

5) For negative *value* **multiplicand** integers and a negative **multiplier**:

Example: **-** $^{-}5$ **-** $^{-}5$ **-** $^{-}5$ **-** $^{-}5$ = $^{-}5$ **x (-**$^{+}4$) (meaning $^{-}5$ **subtracted** together $^{+}4$ *times*) = $^{+}20$
- $^{-}4$ **-** $^{-}4$ **-** $^{-}4$ **-** $^{-}4$ **-** $^{-}4$ = $^{-}4$ **x (-**$^{+}5$) (meaning $^{-}4$ **subtracted** together $^{+}5$ *times*) = $^{+}20$

Normally written: $^{-}5$ **x** $^{-}4$ = 20 and $^{-}4$ **x** $^{-}5$ = 20
~~~~~

6) For a zero *value* **multiplicand** integer and a negative **multiplier**:

Example:
- 0 **-** 0 **-** 0 **-** 0 **-** 0 **-** 0 **-** 0 = <u>0 **x (-**$^{+}$7)</u> (meaning 0 **subtracted** together $^{+}$7 *times*) = 0

Normally written: 0 **x** $^{-}$7 = 0

And it works!

For the fourth example, if we **subtract** a **positive** value we are decreasing what we have, and so our answer must be **negative** as we are moving to the **left** on our number line.

For the fifth example, if we **subtract** a **negative** value we are decreasing our debt or deficit, and so our answer must be **positive** as we are moving to the **right** on our number line.

This is by far the only satisfying answer I know for why two ***negatives*** multiplied together equal a positive. One ***negative*** is the *mathematical operator,* and the other ***negative*** is a *value*! This is a very important distinction that is never considered when multiplying two ***negative*** numbers together!

We can see for the **operator** (in brackets), how we use both a negative *mathematical operator,* and a positive *value* for how many *times* we **subtract** the positive, negative, or zero *value* of the **multiplicand** together.

~~~~~

Possibilities 1) to 6) are all quite straight forward.

We see how conventionally we simply write, for example 5 **x** 4 = 20, confusing number *values* with the *mathematical operators,* and obscuring what the *mathematical operation* of ***multiplication*** actually implies.

For a full notation using symbols for both *values* and *mathematical operators* we can write:

For example: <u>$^{+}$5 **x (+**$^{+}$4)</u> = **+** $^{+}$5 **+** $^{+}$5 **+** $^{+}$5 **+** $^{+}$5 = $^{+}$20
(meaning $^{+}$5 **added** together $^{+}$4 *times*)
~~~~~

Possibilities 1) to 6) can then be written completely as:

$$^{+/-/0}\textbf{multiplicand} \;\; \textbf{x} \;\; ^{+/-}\textbf{multiplier} \;=\; ^{+/-/0}\textbf{product}$$

We can also write possibilities 1) to 6) as:

$$^{+/-/0}\textbf{multiplicand} \;\; \textbf{x} \;\; \textbf{+/-}^{+}\textbf{operator} \;=\; ^{+/-/0}\textbf{product}$$

The **multiplicand** above is either a *positive, negative or zero* **integer**.

The **+** *(positive)* symbol for the **operator** shows the *mathematical* *operation* of **adding** terms of the **multiplicand** together.

The **-** *(negative)* symbol for the **operator** shows the *mathematical* *operation* of **subtracting** terms of the **multiplicand** together.

The *value* of the **operator** with the positive superscript tells us how many *times* we **add** or **subtract** the **multiplicand** together.

The **product** will be either a *positive, negative or zero* **integer**.

~~~~~

But what if the *value* of our **multiplier** is zero implying neither a ***positive*** *nor* ***negative*** *mathematical operation?*

$$^{+/0-}\textbf{multiplicand} \;\; \textbf{x} \;\; ^{0}\textbf{multiplier} \;=\; ^{?}\textbf{product}$$

We looked briefly at this in the chapter on **Natural numbers**, for example, when we were trying to understand what 7 x 0 meant? 7 x 0 tells us that 7 appears 0 *"times",* which implies there are ***no*** 7's. It is like specifically pointing out the ***absence*** of any 7's.

When there are ***no*** 7's our **multiplier** is neither *adding* nor *subtracting* **multiplicands** together. Yet our **multiplier** is still functioning as a *mathematical operator* telling us there are ***no*** 7's or an ***absence*** of 7's.

Can **zero** as a **multiplier** also function as a ***mathematical operator***, as do the ***positive*** and ***negative*** symbols representing the *mathematical operations* of addition and subtraction?
~~~~~

When multiplying the *value* of a multiplicand by **zero**, we are using neither the positive nor the negative *mathematical operator*. Yet, we are still performing a *mathematical operation* on the multiplicand using **zero**, a *mathematical operation* that is neither positive nor negative.

The integer **zero** will still be written as zero.
The *mathematical operator* ~~zero~~ will be written with a “strikethrough”.
As a *mathematical operator*, **~~zero~~** could also simply be written as a zero with a backslash **⦸**, to show that it is a *mathematical operator*, representing a *mathematical operation.*

In the simplest way, **~~zero~~** *or* **⦸** is a *mathematical operator* that shows the *mathematical operation* of the ***absence*** of a number.

~~~~~

As we saw and notated earlier, for example, $^{+}7 \times 0$ must imply there are ***no*** $^{+}7$’s or an ***absence*** of any $^{+}7$’s.

We can also use a “strikethrough” to show ***absent*** multiplicand integers,
for example, ***no*** $^{+}7$ or **⦸**$^{+}7$ = $^{+}\cancel{7}$

~~~~~

The **+** positive and **-** negative *mathematical operators* are always written in front of the number *value* that they are performing a *mathematical operation* on.

Example: **+** $^{+}5$ **+** $^{+}5$ **+** $^{+}5$ **+** $^{+}5$ = $\underline{^{+}5 \ \mathbf{x}\ (\mathbf{+}^{+}4)}$ (meaning $^{+}5$ **added** together $^{+}4$ *times*) = $^{+}20$

Example: **-** $^{+}5$ **-** $^{+}5$ **-** $^{+}5$ **-** $^{+}5$ = $\underline{^{+}5 \ \mathbf{x}\ (\mathbf{-}^{+}4)}$ (meaning $^{+}5$ **subtracted** together $^{+}4$ *times*) = $^{-}20$

We will do the same for the **~~zero~~** or **⦸** *mathematical operator*.

Example: **⦸**$^{+}7$ = $\underline{^{+}7\ \mathbf{x}\ (⦸^{+}1)}$ (meaning the ***absence*** of $^{+}7$, $^{+}1$ *times*) = $\underline{^{+}\cancel{7}\ \ \mathbf{x}\ ^{+}1}$ = $^{+}\cancel{7}$

~~~~~

If for example, 0 **x** 7 and 7 **x** 0 imply mathematically two different things, then the **Number Property** of **Commutativity**, where $a \times b = b \times a$ no longer holds true.
~~~~~

We may even argue that **Commutativity** doesn't even hold true for non-zero ***multiplication***, as <u>$^{+}5$ ***added*** together $^{+}4$ *times*</u> ($^{+}5$ **x** $^{+}4$) is not exactly the same thing as <u>$^{+}4$ ***added*** together $^{+}5$ *times*</u> ($^{+}4$ **x** $^{+}5$), even though they give the same *sum* or *product.*

~~~~~

It makes sense that the addition or subtraction of many zeros will always give zero,
for example:

+ 0 **+** 0 **+** 0 **+** 0 **+** 0 **+** 0 **+** 0 = <u>0 **x (+**$^{+}7$)</u> (meaning 0 **added** together $^{+}7$ *times*) = 0
and
- 0 **-** 0 **-** 0 **-** 0 **-** 0 **-** 0 **-** 0 = <u>0 **x (-**$^{+}7$)</u> (meaning 0 **subtracted** together $^{+}7$ *times*) = 0

But…to say an ***absent*** number is equal to zero is not correct, as obviously zero, which is ***present***, cannot be the same as something that is ***absent*** (one or more times) from our negative, zero, and positive **integers**!

That is for example, $^{+}7$ **x** 0 ≠ 0 (meaning the ***absence*** of $^{+}7$ is not equal to 0)

This implies that we have not only "***present***" integers, but also "***absent***" integers by the *mathematical operation* of ***multiplying*** the ***multiplicand*** by ***zero***!

~~~~~

<u>$^{+/-/0}$**Multiplicand x** ^{0}**Multiplier = ⊘**$^{+/0/-}$**Product**</u>

A ***zero*** **multiplier**, for positive, negative and zero **multiplicands**, then shows the ***absence*** of the ***multiplicand*** (symbolized by the "**⊘**" *mathematical operator*).

For possibilities 7) to 9), using symbols for both *values* and *mathematical operators,* we write:

7) For a positive *value* **multiplicand** integer and a **zero multiplier**:

Example: ($^{+}7$ **x** 0) = ***no*** $^{+}7$ <u>*or*</u> **⊘**$^{+}7$ = <u>$^{+}7$ **x (⊘**$^{+}1$)</u>
(meaning the ***absence*** of $^{+}7$, $^{+}1$ *time*) = <u>$^{+}$~~7~~ **x** $^{+}1$</u> = $^{+}$~~7~~

Normally written instead as: $^{+}7$ **x** 0 = 0

8) For a negative *value* **multiplicand** integer and a **zero multiplier**:

Example: ($^{-}$7 **x** 0) = ***no*** $^{-}$7 *or* ⦸$^{-}$7 = **$^{-}$7 x (⦸$^{+}$1)**
(meaning the ***absence*** of $^{-}$7, $^{+}$1 *time*) = ~~$^{-}$7~~ **x** $^{+}$1 = ~~$^{-}$7~~

Normally written instead as: $^{-}$7 **x** 0 = 0

Earlier we saw that 0 x 0 is the "***absence*** of 0" which would be "the void".

How can we write this now?

9) For a zero *value* **multiplicand** integer and a **zero multiplier**:

Example: (0 **x** 0) = ***no*** 0 *or* ⦸0 = **0 x (⦸$^{+}$1)**
(meaning the "***absence*** of 0", $^{+}$1 *time*) = ~~0~~ **x** $^{+}$1 = ~~0~~ "the void"!

Normally written instead as: 0 **x** 0 = 0

~~~~~

Possibilities 7) to 9) can then be written completely as:

$^{+/0/-}$**multiplicand x** ^{0}**multiplier** = ⦸ $^{+/-/0}$**product**

We can also write possibilities 7) to 9) as:

$^{+/-/0}$**multiplicand x** ⦸$^{+}$**operator** = ⦸$^{+/-/0}$**product**

The **multiplicand** above is either a *positive, negative or zero* **integer**.

The ⦸ *(absent)* symbol for the **operator** shows the *mathematical operation* of making ***absent*** the **multiplicand**.

The *value* of the **operator** with the positive superscript tells us how many *times* we make ***absent*** the **multiplicand**.

The **product** will be either the ***absence*** of a *positive, negative or zero* **integer**.
~~~~~

~~~~~

The nine possibilities of ***multiplying*** the ***multiplicand*** by the ***multiplier*** can be written then as:

1) to 6) $^{+/-/0}$**multiplicand x** $^{+/-}$**multiplier** = $^{+/-/0}$**product**

7) to 9) $^{+/0/-}$**multiplicand x** ^{0}**multiplier** = **ø** $^{+/0/-}$**product**

We can also write:

1) to 6) $^{+/-/0}$**multiplicand x +/-**$^{+}$**operator** = $^{+/-/0}$**product**

7) to 9) $^{+/-/0}$**multiplicand x ø**$^{+}$**operator** = **ø**$^{+/-/0}$**product**

~~~~~

We can continue the **multiplication** of the ***multiplicand*** with as many additional ***multipliers*** as we like.

We gave the following examples for the *nine* ***multiplicand / multiplier*** possibilities:

1) $^{+}$5 **x** (**+**$^{+}$4) $^{+}$4 **x** (**+**$^{+}$5) (positive multipliers)
2) $^{-}$5 **x** (**+**$^{+}$4) $^{-}$4 **x** (**+**$^{+}$5)
3) 0 **x** (**+**$^{+}$7)

4) $^{+}$5 **x** (**-**$^{+}$4) $^{+}$4 **x** (**-**$^{+}$5) (negative multipliers)
5) $^{-}$5 **x** (**-**$^{+}$4) $^{-}$4 **x** (**-**$^{+}$5)
6) 0 **x** (**-**$^{+}$7)

7) $^{+}$7 **x** (**ø**$^{+}$1) (zero multipliers)
8) $^{-}$7 **x** (**ø**$^{+}$1)
9) 0 **x** (**ø**$^{+}$1)

~~~~~

Additional positive or negative ***multipliers*** just simply multiply possibilities 1) to 6).

Examples for 1): [$^{+}$5 **x** (**+**$^{+}$4)] **x** (**+**$^{+}$2) = **+** [$^{+}$20] **+** [$^{+}$20] = $^{+}$40

[$^{+}$5 **x** (**+**$^{+}$4)] **x** (**-**$^{+}$2) = **-** [$^{+}$20] **-** [$^{+}$20] = $^{-}$40
~~~~~

Examples for 6): $[\ \underline{0\ \mathbf{x}\ (\mathbf{-}^{+}7)}\ ]\ \underline{\mathbf{x}\ (\mathbf{+}^{+}2)} = \mathbf{+}[\ \underline{0}\ ]\ \mathbf{+}[\ \underline{0}\ ] = 0$

$[\ \underline{0\ \mathbf{x}\ (\mathbf{-}^{+}7)}\ ]\ \underline{\mathbf{x}\ (\mathbf{-}^{+}2)} = \mathbf{-}[\ \underline{0}\ ]\ \mathbf{-}[\ \underline{0}\ ] = 0$

~~~~~

Possibilities 7) to 9) however are neither positive nor negative. Can we even add or subtract ***absent*** numbers as we do normal numbers?

For example, does $\mathbf{+}[\boldsymbol{\varnothing}^{+}7]\ \mathbf{+}[\boldsymbol{\varnothing}^{+}7] = [\boldsymbol{\varnothing}^{+}14]$, meaning the (***absence*** of $^{+}7$) *added* together 2 *times* equals the (***absence*** of $^{+}14$)?

It is hard to imagine what it means to ***add*** or ***subtract*** an ***absent*** number or if it is even possible?

Example for 7): $(\underline{^{+}7\ \mathbf{x}\ 0})\ \underline{\mathbf{x}\ (\mathbf{+}^{+}2)} = \mathbf{+}[\boldsymbol{no}\ ^{+}7]\ \mathbf{+}[\boldsymbol{no}\ ^{+}7]$ *or* $\mathbf{+}[\boldsymbol{\varnothing}^{+}7]\ \mathbf{+}[\boldsymbol{\varnothing}^{+}7] = [\boldsymbol{\varnothing}^{+}7]\ \mathbf{x}\ (\mathbf{+}^{+}2)$

(meaning the [***absence*** of $^{+}7$] ***added*** together $^{+}2$ *times*) = $\underline{^{+}\not{7}\ \ \mathbf{x}\ (\mathbf{+}^{+}2)}$

$(\underline{^{+}7\ \mathbf{x}\ 0})\ \underline{\mathbf{x}\ (\mathbf{-}^{+}2)} = \mathbf{-}[\boldsymbol{no}\ ^{+}7]\ \mathbf{-}[\boldsymbol{no}\ ^{+}7]$ *or* $\mathbf{-}[\boldsymbol{\varnothing}^{+}7]\ \mathbf{-}[\boldsymbol{\varnothing}^{+}7] = [\boldsymbol{\varnothing}^{+}7]\ \mathbf{x}\ (\mathbf{-}^{+}2)$

(meaning the [***absence*** of $^{+}7$] ***subtracted*** together $^{+}2$ *times*) = $\underline{^{+}\not{7}\ \ \mathbf{x}\ (\mathbf{-}^{+}2)}$

Example for 8): $(\underline{^{-}7\ \mathbf{x}\ 0})\ \underline{\mathbf{x}\ (\mathbf{+}^{+}2)} = \mathbf{+}[\boldsymbol{no}\ ^{-}7]\ \mathbf{+}[\boldsymbol{no}\ ^{-}7]$ *or* $\mathbf{+}[\boldsymbol{\varnothing}^{-}7]\ \mathbf{+}[\boldsymbol{\varnothing}^{-}7] = [\boldsymbol{\varnothing}^{-}7]\ \mathbf{x}\ (\mathbf{+}^{+}2)$

(meaning the [***absence*** of $^{-}7$] ***added*** together $^{+}2$ *times*) = $\underline{^{-}\not{7}\ \ \mathbf{x}\ (\mathbf{+}^{+}2)}$

$(\underline{^{-}7\ \mathbf{x}\ 0})\ \underline{\mathbf{x}\ (\mathbf{-}^{+}2)} = \mathbf{-}[\boldsymbol{no}\ ^{-}7]\ \mathbf{-}[\boldsymbol{no}\ ^{-}7]$ *or* $\mathbf{-}[\boldsymbol{\varnothing}^{-}7]\ \mathbf{-}[\boldsymbol{\varnothing}^{-}7] = [\boldsymbol{\varnothing}^{-}7]\ \mathbf{x}\ (\mathbf{-}^{+}2)$

(meaning the [***absence*** of $^{-}7$] ***subtracted*** together $^{+}2$ *times*) = $\underline{^{-}\not{7}\ \ \mathbf{x}\ (\mathbf{-}^{+}2)}$

Example for 9): $(\underline{0\ \mathbf{x}\ 0})\ \underline{\mathbf{x}\ (\mathbf{+}^{+}2)} = \mathbf{+}[\boldsymbol{no}\ 0]\ \mathbf{+}[\boldsymbol{no}\ 0]$ *or* $\mathbf{+}[\boldsymbol{\varnothing}\ 0]\ \mathbf{+}[\boldsymbol{\varnothing}\ 0] = [\boldsymbol{\varnothing}\ 0]\ \mathbf{x}\ (\mathbf{+}^{+}2)$

(meaning the [***absence*** of 0] ***added*** together $^{+}2$ *times*) = $\underline{\not{0}\ \mathbf{x}\ (\mathbf{+}^{+}2)}$

= "the void" *added* together *twice*!

$(\underline{0\ \mathbf{x}\ 0})\ \underline{\mathbf{x}\ (\mathbf{-}^{+}2)} = \mathbf{-}[\boldsymbol{no}\ 0]\ \mathbf{-}[\boldsymbol{no}\ 0]$ *or* $\mathbf{-}[\boldsymbol{\varnothing}\ 0]\ \mathbf{-}[\boldsymbol{\varnothing}\ 0] = [\boldsymbol{\varnothing}\ 0]\ \mathbf{x}\ (\mathbf{-}^{+}2)$

(meaning the [***absence*** of 0] ***subtracted*** together $^{+}2$ *times*) = $\underline{\not{0}\ \mathbf{x}\ (\mathbf{-}^{+}2)}$

= "the void" *subtracted* together *twice*!

~~~~~

An additional zero ***multiplier*** will ***absent*** any of the above *nine* possibilities.

Example for 1): [$^{+}5$ **x** (**+**$^{+}4$)] **x** 0 = ***no*** [$^{+}20$] *or* ø [$^{+}20$] = $^{+}20$ **x** (ø$^{+}1$)

(meaning the ***absence*** of $^{+}20$, $^{+}1$ *time*) = $^{+}$~~20~~ **x** $^{+}1$ = $^{+}$~~20~~

Example for 6): [0 **x** (**-**$^{+}7$)] **x** 0 = ***no*** [0] *or* ø [0] = 0 **x** (ø$^{+}1$)

(meaning the ***absence*** of 0, $^{+}1$ *time*) = ~~0~~ **x** $^{+}1$ = ~~0~~

Example for 7): ($^{+}7$ **x** 0) **x** 0 = ***no*** (***no*** $^{+}7$) *or* ø (ø$^{+}7$) = [$^{+}7$ **x** (ø$^{+}1$)] **x** (ø$^{+}1$)

(meaning the ***absence*** $^{+}1$ *time* of the (***absence*** of $^{+}7$, $^{+}1$ *time*)) = [$^{+}$~~7~~ **x** $^{+}1$] **x** (ø$^{+}1$) = ø$^{+}$~~7~~

Example for 8): ($^{-}7$ **x** 0) **x** 0 = ***no*** (***no*** $^{-}7$) *or* ø (ø$^{-}7$) = [$^{-}7$ **x** (ø$^{+}1$)] **x** (ø $^{+}1$)

(meaning the ***absence*** $^{+}1$ *time* of the (***absence*** of $^{-}7$, $^{+}1$ *time*)) = [$^{-}$~~7~~ **x** $^{+}1$] **x** (ø$^{+}1$) = ø$^{-}$~~7~~

Earlier we saw that 0 x 0 is the "***absence*** of 0" which would be "the void",
and that, (0 x 0) x 0 would be the ***absence*** of "the void",
or in other words, the "***absence*** of the (***absence*** of 0)", and so on.

Example for 9): (0 **x** 0) **x** 0 = ***no*** (***no*** 0) *or* ø (ø 0) = [0 **x** (ø $^{+}1$)] **x** (ø $^{+}1$)

(meaning the ***absence*** $^{+}1$ *time* of the (***absence*** of 0, $^{+}1$ *time*)) = [~~0~~ **x** $^{+}1$] **x** (ø$^{+}1$)) = ø~~0~~

= (the *absence* of "the void"!)

In simplest terms then, ø~~0~~ means the *mathematical operator* ø**,** executing the *mathematical operation of* ***absence***, upon the (***absence*** or *zero*).

Can we simplify [0 **x** (ø$^{+}1$)] **x** (ø$^{+}1$), meaning [(0 ***absented*** $^{+}1$ *time) **absented*** $^{+}1$ *time*],
to [0 **x** (ø$^{+}1$)$^{\mathbf{+2}}$], meaning [0 (***absented*** $^{+}1$ *time*)$^{\textbf{squared}}$] ?

In simplest terms this would be equal to: ø$^{\mathbf{+2}}$ 0

We can also see that: ø$^{\mathbf{+2}}$ 0 = ø ~~0~~

We can continue on with the (***absence*** of the ***absence*** of the ***absence*** of an ***integer***), and any other possible combination for ***multiplication***, and so on, forever!

~~~~~

Finally, the number of "*times*" we use a *mathematical operator is* always *positive,* for example: **+** $^{+}5$ **+** $^{+}5$ **+** $^{+}5$ **+** $^{+}5$ = $^{+}5$ **x (+**$^{+}4$) (meaning $^{+}5$ **added** together $^{+}4$ *times*) = $^{+}20$

The *positive "times"* value of the *mathematical operator* means, we have repeated a *mathematical operation* one or more *times*.

In the above example $^{+}4$ as the *positive* value of 4 makes sense. $^{0}4$ however makes no sense as the *value* of the number 4 can only be *positive or negative.* A zero superscript only makes sense for $^{0}0$, even though it is redundant.

If we write: $^{+}5$ **x (+** $^{0}0$) or $^{+}5$ **x (-** $^{0}0$) or $^{+}5$ **x (⊘** $^{0}0$) it means we are repeating the *mathematical operation* of *addition, subtraction or absence* on the ***multiplicand*** zero *"times".*

This is not the same as $^{+}5$ **x (⊘**$^{+}1$) (meaning the ***absence*** of $^{+}5$, $^{+}1$ *time*).

$^{+}5$ **x (+** $^{0}0$) or $^{+}5$ **x (-** $^{0}0$) or $^{+}5$ **x (⊘** $^{0}0$) means we are affecting no *mathematical operation*, whether *positive, negative or absent* upon the $^{+}5$ ***multiplicand***.
For example, $^{+}5$ **x (+** $^{0}0$) means we are ***adding*** $^{+}5$ to itself *zero times,* which could be another way of saying that the $^{+}5$ ***multiplicand*** doesn't even exist!
This is not the same thing as saying that $^{+}5$ is ***absent***!

And finally, is there any meaning to, for example: $^{+}5$ **x (+**$^{-}4$) (meaning $^{+}5$ **added** together $^{-}4$ *times*)?

Can we repeat a *mathematical operation* upon a ***multiplicand*** a *negative* number of *times,* in a strange way and mirror reality, "opposite" to the way we repeat the *mathematical operation* a *positive* number of *times*?

And what would that even mean?

A *positive "times" value* means we are affecting a *mathematical operation* one or more *times.*

A *zero "times" value* means we are affecting a *mathematical operation* zero *times.*

A *negative "times" value* means we are affecting a *mathematical operation* in a strange way, in an opposite mirror reality to a *positive mathematical operation*!
~~~~~

~~~~~

Summary for Integer Multiplication

It is hard to believe that we can write more than ten pages trying to logically understand ***multiplication*** for ***integers***!

While sound *logic* can lead us here it can also melt our mind!!

Of most importance is distinguishing the *value* of an ***integer*** from a *mathematical operator,* both of which use *positive* and *negative* symbols. There is no proper understanding of the short form of ***multiplication***, without realizing that the *positive or negative value* of an ***integer*** is not the same thing as a *positive or negative mathematical operator,* which represents a *mathematical operation. T*his is most clearly seen when we ***multiply*** a *negative* ***multiplicand*** by a *negative* ***multiplier***.

We keep the normal use of *positive and negative* symbols as *mathematical operators. Positive* and *negative* symbols for the *value* of an ***integer*** are shown in superscript.

For ***integers***, the *value* of the ***multiplicand*** and ***multiplier*** can be either *positive, negative or zero.* We run into a logical problem when the ***multiplier*** is zero, as a zero ***multiplier*** implies that there is ***no multiplicand***, or in other words, that the ***multiplicand*** is ***absent.***

And even though the logic holds, the idea of an ***absent*** number is not a common or even known concept. This would mean that ***integers*** consist of not only "***present***" *positive, negative and zero* numbers, but also "***absent***" *positive, negative and zero* numbers!

Absent numbers go against the **Number Property** of **Commutativity**, where $a \times b = b \times a$, for example 0 **x** 7 = 7 **x** 0. With ***absent*** numbers we are actually saying that 0 **x** 7 ≠ 7 **x** 0, that 0 **x** 7 is not the same thing as 7 **x** 0.

To challenge the validity of ***absent*** numbers we must first undo the *logic* that brought us to the idea of ***absent*** numbers. Whether ***absent*** numbers may be useful, who knows?
~~~~~

~~~~~

Division

(In Preparation)
~~~~~